Activity - Solutions

1. Calculate the modulus and the argument of the solutions to finding the square root of unity and represent these solutions on an Argand diagram. What is the sum of these roots?

The solutions to finding the square roots of unity are \( z_1 = 1 \) and \( z_2 = -1 \). As shown in the Argand diagram.

From the diagram we can see that \( |z_1| = 1 \) and similarly, \( |z_2| = 1 \). Additional, we can see that \( \arg(z_1) = 0 \) and \( \arg(z_2) = \pi \).

Summing these roots gives \( z_1 + z_2 = 1 + -1 = 0 \).

2. Calculate the modulus and the argument of the solutions to finding the cube root of unity and represent these solutions on an Argand diagram. What is the sum of these roots?

The solutions to finding the cube roots of unity are \( z_1 = 1 \), \( z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i \) and \( z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2} i \).

From the diagram we can see that \( |z_1| = 1 \) and \( \arg(z_1) = 0 \).
The Roots of Unity – Activity Solutions

The modulus of $z_2$ and $z_3$ is,

$$|z_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1 \quad \text{and} \quad |z_3| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1$$

The principle argument of $z_2$ and $z_3$ is

$$\arg(z_2), \quad \tan \theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \Rightarrow \quad \arg(z_2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\arg(z_3), \quad \tan \theta = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \quad \Rightarrow \quad \arg(z_3) = -(\pi - \frac{\pi}{3}) = -\frac{2\pi}{3}$$

Summing these roots gives $z_1 + z_2 + z_3 + z_4 = 1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$

3. Use the function $f(x) = z^4 - 1$ to find the fourth roots of unity. Calculate the modulus and the argument of these solutions and represent them on an Argand diagram. What is the sum of these roots? Comment on any patterns that you have noticed.

We can use the function $f(z) = z^4 - 1$ to find the fourth roots of 1.

$$f(z) = 0$$

$$z^4 - 1 = 0$$

$$z^4 = 1$$

$$\therefore \ z^2 = \pm 1$$

When $z^2 = 1$, we have the solutions $z = \pm 1$ and when $z^2 = -1$, we have the solutions $z = \pm i$

Thus, the fourth roots of unity are,

$$z_1 = 1, \ z_2 = i, \ z_3 = -1 \quad \text{and} \quad z_4 = -i$$

and are represented in the following Argand diagram.
From the diagram we can see that $|z_1| = |z_2| = |z_3| = |z_4| = 1$ and that $\text{arg}(z_1) = 0$, $\text{arg}(z_2) = \frac{\pi}{2}$, $\text{arg}(z_3) = \pi$ and $\text{arg}(z_4) = \frac{3\pi}{2}$.

Summing these roots gives $z_1 + z_2 + z_3 + z_4 = 1 + i + 1 - 1 - i = 0$

Some patterns that you may have noticed are:

- The modulus to the $n^{th}$ roots of unity is always equal to 1.
- The argument the $n^{th}$ roots of unity is always $\frac{2m\pi}{n}$, where $m \in \mathbb{Z}$
- The sum of the $n^{th}$ roots of unity is zero.